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Conformal Groups and Related Symmetries Physical Results and Mathematical Background

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DE - SITTER REPRESENTATIONS AND THE PARTICLE CONCEPT, STUDIED IN AN UR-THEORETICAL COSMOLOGICAL MODEL x)

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Abstract: The theory of urs (basic two-valued observables) is used to describe particles in cosmic space-time. Cosmic position space is described as S^3, interpreted as a homogeneous space of SU(2). An expanding model of the universe is locally approximated by de Sitter spaces. Irreducible representations of the de Sitter group are explicitly constructed in ur theory. From these, Poincaré group representations in Minkowski space with well-defined rest mass are deduced by a special rule of contraction.

1. Ur - Theory and Cosmology

We use the terms a b s t r a c t q u a n t u m t h e o r y for the universal laws of quantum theory in Hilbert space, and c o n c r e t e q u a n t u m t h e o r y for the description of objects as they really exist in the world. /1/ Abstract quantum theory includes the universal law of dynamics: the time dependence of states is described by a one-parameter unitary transformation group in the Hilbert space. Concrete quantum theory comprises the existence of particles in a 3,1-dimensional space-time with relativistically invariant interaction laws. We call u r h y p o t h e s i s the assumption that all state spaces occuring in concrete quantum theory can be resolved into tensor products of two-dimensional complex vector

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spaces which are defined by one unique type of observable, called Ur - Alternative (e.g. original alternative) in German; the abstract object corresponding to this alternative is called the ur. We call u r t h e o r y the study of the premisses and consequences of the ur hypothesis. (/1/ ch. 9 and 10)

The ur theory will have to test two conjectures:

The s u f f i c i e n c y c o n j e c t u r e (SC): The ur hypothesis is sufficient for deducing the complete concrete quantum theory from abstract quantum theory.

The t r i v i a l i t y c o n j e c t u r e (TC): The ur hypothesis is trivial in the sense of being a necessary concequence of those postulates from which abstract quantum theory itself can be reconstructed.

SC, the sufficiency of the ur hypothesis, may seem to be a very daring assumption. In order to confirm it we would have to deduce from the quantum theory of an arbitrary number of urs

a) the existence of a 3,1-dimensional space-time

b) the existence and properties of all known particles and fields.

We suppose to have solved problem a) in /1/ chapter 9, as far as space can be described as flat or constantly curved. The basic idea is to interpret a symmetry group of the ur, SU(2), as defining a real 3-space in which it acts as an SO(3), and which is hypothetically treated in the ur theory as the position space of physics. The solution of problem b) cannot be easier than a unified theory of elementary particles (/1/, chapter 10). The present paper describes a special model, not contained in /1/, in which we can formulate an ur-theoretical approach towards problem b).

In any new fundamental theory there occurs a reversal of the historical order of some arguments. We use two well-known examples which will turn out to be relevant again for ur theory.

Astronomy, the most ancient exact natural science, was from antiquity down to Kepler a morphological theory of planetary orbits, be it around the earth or around the sun, completed by the morphological cosmology of a finite spherical universe. General laws prescribed the mathematical form of the orbits as built from circles or, finally, as ellipses. These shapes were different from those of terrestrical motions. Newton's mechanics was not a "great", i. e. additive, but a "radical", i. e. reductive unification. In astronomy it

permitted for the first time to ask and answer the question why and to what approximation planets should have mathematically well-defined orbits at all. Historically the orbits were the way towards general mechanics; in the new theory mechanics was the reason why there had to be orbits.

Similarly, the quantum theory of the atom was a radical unification of mechanics and chemistry. Bohr's correspondence principle presupposed the good approximate macroscopic validity of classical physics and paved the way towards a consistent quantum theory. Quantum mechanics reversed the argument and explained classical mechanics as its limiting case.

Ur theory again pressuposes the good approximate validity of two earlier concepts: of the v i s i b l e u n i v e r s e and of p a r t i c l e s . The high degree of homogeneity and the systematic expansion of the visible universe has permitted to treat it approximately as one large physical object. Its history is described in the semi-empirical , semi-speculative science of cosmology. This science presupposes general relativity which was conceived as a theory of a locally defined field. On the other hand the concepts of mass-point particles and/or localisable fields were presupposed by most models of elementary particle theories; a string in a high-dimensional space is no more than a generalisation of the particle concept. An additive unification of the concepts of universe and particle has begun to exist in the description of the earlier phases of cosmic expansion. Both concepts presuppose the validity of the concept of space-time.

Since the ur theory claims to derive position space from the quantum state space of the binary alternative, it is essentially a radical unification of cosmology and particle theory. A single ur, containing no more than one bit of information, cannot possibly be localized in the universe. The simplest model of cosmic space in the ur theory is the largest homogeneous space of the group SU(2), that is the group itself, considered as a topological and metrical space. It is the S^3, the position part of an Einstein cosmos. In this space, one ur can be described as a spinor wave-function with a wavelength equal to the diameter of the universe (/1/, chapter 9, section 3b). If we assume this diameter to be 10^{27}cm, a particle can be localized down to the Compton wavelength of the electron by superposing 10^{37} ur wavefunctions. Thus the ur is essentially cosmic. The accuracy of the measurement of a small distance is limited by the available number of urs.

In this theory, space is not an objective ultimate entity like Newton's or Einstein's spaces. Its coordinatisation as S^3 is done by the group parameters of SU(2) which are no quantum observables. As far as its properties can be observed, it is rather a "surface" of the high-dimensional quantum state space of a large number of urs. Quantum theory is immensely richer in information than any classical theory in space-time.

On the other side, the concept of particle equally loses its apparent evidence. If a particle "consists" of 10^{37} urs, why should these stick together? This problem should not surprise us. The history of atomism teaches that nearly every particle which was considered elementary turned out to be composite sooner or later. Ur theory seems to be the most radical possible form of atomism; there is no smaller meaningful alternative than yes - no. Hence we may expect all objects to be divisible into urs in principible. This division is no longer spatial, but informationial. The question then is: what is the dynamics of the urs; how does it motivate them to keep together? This can be subdivided into two succesive questions:

1) How to describe the inertial motion of a free particle?
2) How to describe interaction?

The present paper is confined to question 1). The answer is given in principle by Wigner's definition: The state space of a free pointlike particle is the representation space of an irreducible representation of the Poincaré group. Thus, if we can construct such representations by urs, they will permit an interpretation as particles.

The problem is that ur theory does not yet fully determine the relevant relativistic group. In order to understand this problem we turn to TC, the conjecture that ur theory is trivial (/1/, chapter 9, section 2b). It is indeed trivial as far as we leave dynamics aside. It is logically trivial that any n-fold alternative can be represented within the Cartesian product of k binary alternatives with $2^k \geq n$. It is mathematically true that an n-dimensional vector space can be represented within the tensor product of k two-dimensional vector spaces. If n is countably infinite, so will be k . This decomposition is not unique; there are many different possibilities of defining the ur. The problem is wether the law of dynamics keeps all these differently defined urs or some of them invariant in time, such that they can be considered as physical objects (or, rather, "subobjects").

Certainly the ur hypothesis is not trivial in full abstract quantum theory which permits any self-adjoint operator as an Hamiltonian. In /1/, ch. 9, sec. 2b, we try to narrow down the basic postulates of quantum theory so as to make TC a necessary concequence. In the present paper we follow a different path, by way of a cosmological model.

We assume S^3, as defined by the symmetry group of the ur, to be a parametrisation of the cosmic position space. S^3 is compact. Hence we seem to have assumed a finite universe. As long as we may assume that the information content, hence the number of urs, if the universe is finite, a compact position space is indeed a natural description. (We should never forget that in ur theory space is not a basic entity, but, only a way of describing a quantum world, hence perhaps to some extent conventional.) If, however, we assume an infinite number of urs, we must represent the quantum state of the universe in an infinite-dimensional Hilbert space in which noncompact groups possess unitary representations. Then we can use unbounded world models.

Yet, in an infinite-dimensional Hilbert space we must distinguish between actual and virtual urs, i.e. between alternatives that can be decided, given a real situation, and alternatives that might only be decided by producing a different situation. A free particle in flat position space is an example. A discrete basis of its wave functions is given by all eigenfunctions of the total angular momentum, j, and of one of its components, m. The angular momentum is defined with respect to some position in space; let this be the observer's position. Then there will be an upper bound j_{max} for those wave functions which the observer will be able actually to observe; for $j > j_{max}$ the value of ψ in the volume accessible to the observer will, for him, be practically indistinguishable from zero. That means that this observer will only make use of a finite-dimensional part of the Hilbert space of the particle; a part which can be decomposed into the state spaces of a finite number of urs. If he wants to know more about the particle, he must move to another place, finding an additional finite number of decisions; and so on. The full representations of non-compact groups like spatial translations or Lorentz boosts is always done by virtual urs; we cannot actually walk indefinitely into space or accelerate a particle indefinitely.

For a cosmological working model /2/ we choose a cosmological (absolute) time coordinate t and a time-dependent total number $N_u(t)$ of urs in the universe. N_u is supposed to increase monotonously with t : the number of possible decisions in the world increases steadily with time. N_u will be a measure of the volume of the cosmic space, as measured by elementary particles. Thus our assumption describes the expansion of the universe. The semantic consistency of the model will only admit a test when we shall have understood how particles can be described in such a universe.

2. Particles

The concept of a pointlike particle is historicaly an abstraction from the concept of a body, or of its center of mass, neglecting the body's extension or inner dynamics. In the ur theory, this concept must be derived from more basic concepts. Wigner derived the free particle from the representation theory of the Poincaré group. The Wigner particles are characterised by two numbers: the spin s, and the mass m. In the ur theory, particles with arbitrary spin can be represented (/1/, ch. 9, sec. 3e). The determination of m remains an unresolved problem (/1/, ch. 10, sec. 6d).

In the ur-theoretical context it is plausible that the rest masses of real particles are cosmologically determined. A possible consideration might be the following: We measure cosmic dimensions by ponderable matter (and with the help of light). The mass of ponderable matter is mainly concentrated in nucleons. Let λ be the Compton wavelength of the nucleon, R_u the radius of the universe. Assume the number N_u of urs in the universe to be the volume of the universe, measured in nuclear volumes:

$$N_u \approx R_u^3 / \lambda^3 \tag{1}$$

In order to localize a nucleon in 3 dimensions we need

$$\gamma \approx 3 R_u/\lambda \approx N^{1/3} \tag{2}$$

urs. If we assume this to be the number of urs "contained" in the nucleon, there would have to be $N^{2/3}$ nucleons in the

world. With $N^{1/3} \approx 10^{40}$, this gives 10^{80} nucleons, not too far from the estimated empirical number. The real task of the theory would be to explain this "condensation" of urs by statistical considerations.

Our present aim is more limited. We search for a precise mathematical description of particles in our cosmological model. Wigner's construction presupposes a Minkowski space. It would seem natural at any space-time point to choose the locally tangential Minkowski space. This is what we will finally do. But we shall insert a locally approximating de Sitter space between the cosmological model and the Minkowski space. The reason for this intercalation is that a de Sitter space combines two properties, none of which ought to be lost. It contains an S^3 as its spatial part; by approximating the world model this can be identified with the S^3 prevailing in the model at the respective cosmological time t . We need this compact spatial volume in order to perform the program of determining the rest mass of the particle. And on the other hand it possesses a 10-parameter symmetry group which will permit us to define particles by the Wigner method which then can be translated into the usual Minkowski description by the procedure of contraction.

The free particles thus defined will then be the starting material for a theory of interaction (/1/, ch. 10) including the transition to the Riemannian space-time of general relativity (/1/, ch. 10, sec. 7; /2/).

3. How to build Particles out of Urs

The state space of an ur is C^2. The norm is conserved by $SU(2) \times U(1)$, and by complex conjugation. The latter can, according to Castell /3/, be linearly represented by introducing anti-urs, represented together with the urs in a common C^4. The state space for n urs is then the tensor product $C^{4^n} = \bigotimes_1^n C^4$. All states of any number of urs are then contained in the "tensor space" $T = \bigoplus_{n=0}^{\infty} C^{4^n}$. C^1 is the vacuum. Let r_i (i = 1...4) be a basis in C^4. Then the monomials

$r_{i_1}\dots r_{i_n}$ form a basis in C^{4^n} with

$$\langle r_{i_1}\dots r_{i_n} \mid r_{k_1}\dots r_{k_m}\rangle = \delta_{nm}\cdot\delta_{i_1k_1}\dots\delta_{i_nk_n} \qquad (3)$$

Now we define "pick-" and "stuff-" operators R_r and S_r resp. (see /1/, ch. 10, sec. 2b) by their action on the basis monomials:

$$S_r\, r_1\dots r_n = r r_1\dots r_n + r_1 r\dots r_n + \dots + r_1\dots r r_n + r_1\dots r_n r \qquad (4)$$

$$R_r\, r_1\dots r_n = r_1\dots \cancel{r_{i_1}}\dots r_n + r_1\dots \cancel{r_{i_2}}\dots r_n + \dots + r_1\dots \cancel{r_{i_j}}\dots r_n \qquad (5)$$

with $r_{i_k} = r$ and $r_k \neq r$ for all the other indices; $\cancel{r_{i_j}}$ means omit this vector from the monomial. With respect to the scalar product (3) R_r and S_r are adjoint operators: $R_r^+ = S_r$

Further on we define "trucking" - operators t_{rs} by

$$t_{rs}\, r_1\dots s_1\dots s_k\dots r_n = r_1\dots r\dots s_k\dots r_n + \dots + r_1\dots s_1\dots r\dots r_n \qquad (6)$$

for $s_i \equiv s$ and $r_i \neq s$

The operator t_{rr} only multiplies a monom with the number of vectors r contained in it.

The following commutation relations hold

$$\begin{aligned} [S_i, S_j] &= [R_i, R_j] = 0 & [R_r, t_{st}] &= + R_t\delta_{rs} \\ [R_r, S_s] &= t_{sr} \quad \text{for } s \neq r & [S_r, t_{st}] &= - S_s\delta_{rt} \\ [R_r, S_r] &= t_{rr} + \widehat{(n+1)} \end{aligned} \qquad (7')$$

$\hat{n}$ denotes the operator multiplying a monom with the number of its factors.

Castell has shown how the conformal group SO(4,2) can be represented in the subspace $\bar{T}$ of T which consists of the symmetric tensors only. The operators acting in $\bar{T}$ which are used in these representations are

$$a_r^+ = S_r/\sqrt{n+1} \quad ; \quad a_r = R_r/\sqrt{n} \qquad (8)$$

with the canonical commutation relations

$$[a_r, a_s^+] = \delta_{rs} \quad ; \quad [a_r, a_s] = [a_r^+, a_s^+] = 0 \qquad (9)$$

They correspond to Bose statistics for the urs. In these representations, $s = (n_1 + n_2 - n_3 - n_4)/2$ with n_r designing the number of urs in state r, is the Casimir operator of SO(4,2) which describes the helicity of the respective particle; the whole space $\bar{T}$ contains just one representation for each value of s, describing a massless particle.

If we want to describe many-particle systems and particles with non-zero rest mass we must make use of nonsymmetric tensors. This raises the question wether urs are individually distinguishable. In principle one would prefer to assume them to be indistinguishable, since their distinction would be an additional alternative, not in accord with SC. If TC were correct, the version of the ur hypothesis turning out to be trivial would decide the question. We have so far considered two alternative answers.

The most general statistics for indistinguishable urs is parastatistics, especially para-Bose statistics (/1/, ch. 10, sec. 2d). It acts on a larger subspace $\tilde{T}$ of T which contains in every C^{4^n} one representative of every irreducible representation of the permutation group S_n, i.e. for every Young standard scheme. It permits representations of SO(4,2) and its subgroups with finite rest mass.

The present paper presents an other possibility. It works, in principle, in the full tensor space T. R_r and S_r are defined everywhere in T. If, following a proposal by Drühl, we define

$$R_r = - t_{or} \quad ; \quad S_r = t_{ro} \quad ; \quad \widehat{(n+1)} = - t_{oo} \;, \qquad (10)$$

then the relations (7') can be summarized into

$$[t_{rs}, t_{tu}] = t_{ru}\delta_{st} - t_{ts}\delta_{ru} \qquad (7)$$

Commutation relations for arbitrary powers of these operators are given in the appendix.

As a "ground state" we define a normed vector Ω_N in T with

$$R_r\,\Omega_N = 0 \;;\; t_{rs}\,\Omega_N = 0 \quad \text{for } r \neq s \quad ;\; -t_{oo}\,\Omega_N = (N+1)\,\Omega_N \qquad (11)$$

Ω_0 is the regular vacuum, a general ground state is given by

$$\Omega_{4n} = (4!)^{-n/2} \sum_{P_i} (-1)^{T_1} P_1(r_1r_2r_3r_4)\dots(-1)^{T_n} P_n(r_1r_2r_3r_4) \qquad (12)$$

P_k denotes a permutation of the quadruple of basis vectors and T_k counts the transpositions in it. The sum goes over all possible permutations in all the quadruples.

Over a ground state the stuff-operators generate a linear subspace. Its orthonormal basis vectors are given by

$$|l_1l_2l_3l_4\rangle_N = \sqrt{\frac{(N+n)!}{(N+n+L)!\,l_1!\,l_2!\,l_3!\,l_4!}}\; S_1^{l_1} S_2^{l_2} S_3^{l_3} S_4^{l_4}\, \Omega_N \qquad (13)$$

with $N = 4n$, $L = l_1+l_2+l_3+l_4$

On $|l_1l_2l_3l_4\rangle_N \stackrel{!}{=} |L\rangle_N$ the pick- and stuff-operators act like

$$\begin{aligned} R_j\,|l_i,l_j,l_k,l_l\rangle_N &= \sqrt{l_j}\,\sqrt{N+n+L}\;|l_i,l_j-1,l_k,l_l\rangle_N \\ S_j\,|l_i,l_j,l_k,l_l\rangle_N &= \sqrt{l_j+1}\,\sqrt{N+n+L+1}\;|l_i,l_j+1,l_k,l_l\rangle_N \end{aligned} \qquad (14)$$

$$t_{ji}|l_i,l_j,l_k,l_l\rangle_N = \sqrt{l_j+1}\,\sqrt{l_i}\;|l_i-1,l_j+1,l_k,l_l\rangle_N$$

$$t_{jj}\;|L\rangle_N = (n+l_j)\;|L\rangle_N \qquad -t_{oo}\;|L\rangle_N = (N+L+1)\;|L\rangle_N$$

4. De Sitter representation for a given ground state

Given a ground state Ω_N, then, by modified pick- and stuff-operators, an irreducible unitary representation ν_{r,N^2} of the de Sitter-group /4/ can be constructed such that for different particles the ratio of their numbers of urs in the ground state corresponds to their mass ratio. If the spin r of the particle is not zero then we have (2r+1) vectors with minimal ur number which in this case is N + 2r.

Let the indicees a, b be equal to 1 or 2 and c, d to 3 or 4. Now we define the following operators

$$S_{ac}|L\rangle_N = S_{ca}|L\rangle_N = \sqrt{\frac{(L/2+1-r)(L/2+2+r)}{(l_1+l_2+1)(l_1+l_2+2)(l_3+l_4+1)(l_3+l_4+2)}}\;\cdot$$
$$\cdot\sqrt{\frac{(L/2+1)(L/2+2)+N^2}{(N+n+L+1)(N+n+L+2)}}\cdot S_a\cdot S_c|L\rangle_N \qquad (15)$$

$$R_{ac}|L\rangle_N = R_{ca}|L\rangle_N = \sqrt{\frac{(L/2-r)(L/2+1+r)}{(l_1+l_2)(l_1+l_2+1)(l_3+l_4)(l_3+l_4+1)}}\;\cdot$$
$$\cdot\sqrt{\frac{L/2\,(L/2+1)+N^2}{(N+n+L)(N+n+L+1)}}\cdot R_a\cdot R_c|L\rangle_N \qquad (16)$$

$$T_{ac}|L\rangle_N = \sqrt{\frac{((l_3+l_4-l_1-l_2)/2+r)((l_1+l_2-l_3-l_4)/2+1+r)}{(l_3+l_4)(l_3+l_4+1)(l_1+l_2+1)(l_1+l_2+2)}}\;\cdot \qquad (17)$$
$$\cdot\;1/2\sqrt{(l_3+l_4-l_1-l_2)(2+l_3+l_4-l_1-l_2)+4N^2}\;t_{ac}|L\rangle_N$$

$$T_{ca}|L\rangle_N = \sqrt{\frac{((l_1+l_2-l_3-l_4)/2+r)((l_3+l_4-l_1-l_2)/2+1+r)}{(l_1+l_2)(l_1+l_2+1)(l_3+l_4+1)(l_3+l_4+2)}}\;\cdot \qquad (18)$$
$$\cdot\;1/2\cdot\sqrt{(l_1+l_2-l_3-l_4)(2+l_1+l_2-l_3-l_4)+4N^2}\;t_{ca}|L\rangle_N$$

The generators of the wanted unitary irreducible representation of the SO(4,1) are (19)

$$M_1 = (t_{12}+t_{21}+t_{34}+t_{43})/2 \qquad P_1 = (t_{12}+t_{21}-t_{34}-t_{43})/2$$
$$M_2 = -i(t_{12}-t_{21}+t_{34}-t_{43})/2 \qquad P_2 = -i(t_{12}-t_{21}-t_{34}+t_{43})/2$$

$$M_3 = (t_{11}-t_{22}+t_{33}-t_{44})/2 \qquad P_3 = (t_{11}-t_{22}-t_{33}+t_{44})/2$$

These six operators form the SO(4) - subgroup and preserve the number of urs and of antiurs.

$$P_0 = (S_{14}-S_{32}+R_{14}-R_{32}+T_{31}+T_{13}+T_{42}+T_{24})/2 \qquad (20)$$
$$N_1 = -i(S_{13}-S_{24}+R_{24}-R_{13}+T_{32}-T_{23}+T_{41}-T_{14})/2$$
$$N_2 = -(S_{13}+S_{24}+R_{24}+R_{13}+T_{32}+T_{23}-T_{41}-T_{14})/2$$
$$N_3 = i(S_{14}+S_{23}-R_{14}-R_{23}+T_{13}-T_{31}-T_{24}+T_{42})/2$$

The Casimir-operators for this representation are

$$C_2 = P_0^2 - P_1^2 - P_2^2 - P_3^2 + N_1^2 + N_2^2 + N_3^2 - M_1^2 - M_2^2 - M_3^2 \qquad (21)$$

$$C_4 = (\vec{M}\cdot\vec{P})^2 - (P_0\,\vec{M} - \vec{P}\times\vec{N})^2 - (\vec{M}\cdot\vec{N})^2 \qquad (22)$$

with the eigenvalue equations

$$C_2\;|l_1,l_2,l_3,l_4\rangle_N = (N^2 - r\,(r+1)+2)\,|l_1,l_2,l_3,l_4\rangle_N \qquad (23)$$

$$C_4\;|l_1,l_2,l_3,l_4\rangle_N = -N^2\,r\,(r+1)\,|l_1,l_2,l_3,l_4\rangle_N \qquad (24)$$

written in the pick- and stuff-operators C_2 has the form

$$\begin{aligned}2\,C_2 = {} & S_{14}R_{14} + S_{13}R_{13} + S_{24}R_{24} + S_{23}R_{23} - (t_{11}-t_{22})^2 + \qquad (25)\\ & + R_{14}S_{14} + R_{13}S_{13} + R_{24}S_{24} + R_{23}S_{23} - (t_{33}-t_{44})^2 + \\ & + T_{13}T_{31} + T_{14}T_{41} + T_{23}T_{32} + T_{24}T_{42} - 2(t_{12}t_{21}+t_{21}t_{12}) \\ & + T_{31}T_{13} + T_{41}T_{14} + T_{32}T_{23} + T_{42}T_{24} - 2(t_{34}t_{43}+t_{43}t_{34})\end{aligned}$$

We define (for i, k, l, m mutually different)

$$(t_{ii}+t_{kk}-t_{ll}-t_{mm})T_{ik} + 2t_{il}T_{lk} + 2t_{mk}T_{im}$$
$$\doteq (\hat{l}_1+\hat{l}_2+\hat{l}_3+\hat{l}_4+2)T_{ik} \qquad (26)$$

Then C_4 can be written as

$$\begin{aligned}8C_4 = {} & -(t_{11}+t_{22}+t_{33}+t_{44})^2\left(\{S_{13},R_{13}\}+\{S_{14},R_{14}\}+\{S_{23},R_{23}\}+\{S_{24},R_{24}\}\right) \\ & -(\hat{l}_1+\hat{l}_2+\hat{l}_3+\hat{l}_4+2)^2\left(\{T_{13},T_{31}\}+\{T_{14},T_{41}\}+\{T_{23},T_{32}\}+\{T_{24},T_{42}\}\right) \\ & + 4\,\{t_{12},t_{21}\} - 4\,\{t_{34},t_{43}\} + 2(t_{11}-t_{22})^2 - 2(t_{33}-t_{44})^2 \qquad (27)\end{aligned}$$

In the spin-zero-case there is $r = 0$ and $l_1+l_2 = l_3+l_4$, so all T_{ac} and T_{ca} vanish. For half-integer spin the representations are unitary only for $N^2 \geq 1/4$. Representations with integer spin are unitary also for $N^2 = 0$, but then they decompose into a direct sum of tree irreducible representations of the so-called discrete series $\pi^{\pm}_{r,q}$:

$$\lim_{N^2\to 0}\;\nu_{r,N^2} = \pi^{+}_{r,1} + \pi_{r,0} + \pi^{-}_{r,1} \qquad (28)$$

Castell's massless particles[3] all belong to the representations $\pi^{+}_{r,r}$ and $\pi^{-}_{r,r}$. In the limit (28) only the photon representations are of this type.

5. Transition to the Poincaré Group

We have introduced de Sitter space as an approximation to the cosmological model in order to interpret states from the tensor space of urs as states of a particle in de Sitter space. The particle was defined by its minimal ur number N. It turned out that our operators R and S defined irreducible representations of the de Sitter group characterized by the number N^2. These representations are localisable on the S^3 as considered as the position space in the de Sitter world[5]. To the degree to which we can neglegt the curvature of this S^3, hence approximate it by a flat space, or the de Sitter world by a Minkowski world, such a state can be considered as a localized state in a representation of the Poncaré group P . We shall consider the resulting Poincaré representation as the Wigner description of a free particle.

The transition from the de Sitter representation to the Poincaré representation is achived by a group contraction. It is well known that this contraction can be done in different ways, so as to give the Poincaré particle any value m of its rest mass. We consider N as the quantity in T which corresponds to the rest mass. Hence we shall carry out a contraction such that the ratio N'/N'' of two different particles is transformed into the ratio m'/m'' of their masses.

In the process of contraction a parameter λ which corresponds to the curvature scalar of the de Sitter space goes towards zero. Simultaneously the parameter N^2 which characterises the representation moves towards infinity. The rest mass m in the resulting Poincaré representation is given by

$$m^2 = \lim_{\lambda^2 \to 0\ ;\ N^2 \to \infty} (\lambda^2 N^2) \qquad (29)$$

We need a relation between λ and N in order to fix m. It is sufficient to postulate that this relation should be such that

for two different particles the ratio N'/N'' is kept constant throughout the process of going to the limit; then we will achieve

$$m' / m'' = N' / N'' \qquad (30)$$

We can e.g. abitrarily choose that the Planck-Wheeler mass should be $m_P = 1$ for all time. The number of urs in the Planck-Wheeler particle is $N_u^{1/2}$. If, as our cosmological model assumes, N_u depends on the cosmological time t , the number N of urs in the ground state of a particle whose mass is assumed to have at a given time a fixed value in units of the Planck mass will depend on cosmological time:

$$N / N_u^{1/2} = f(t) \qquad (31)$$

It will, however, depend on the intended theory of rest masses in which way this condition will be specified.

Böhm and Moylan[6] have shown that the representation space of an irreducible representation of SO(4,1) is the direct sum of the representation spaces of two irreducible representations of the Poincaré group, both with positive energy and equal mass m, but different by a charge-like quantum number. So, coming from the de Sitter group, the particle - antiparticle dualism is very natural. In their theory m is not fixed, this is done by our presciption.

We recapitulate our answer to the question 1., how to describe the inertial motion. Locally we have justified the Wigner description in Minkowski space; its empirical success justifies our calling the Minkowski coordinate x_0 the time. Through the local de Sitter space this identification leads back to the local time in the cosmological model. However, with increasing cosmological time t the local Minkowski space is replaced by another one; it is to be assumed that N and m will thus depend on t . This dependence will be determined by the assumed dependence of N_u on t in the cosmological model. Since the actual measurement of time will depend on the functions N(t) and m(t) , t being the assumed cosmological time, it seems possible that the model contains no arbitrary function $N_u(t_m)$, if t_m means the time as locally measured. But this question is further to be studied.

A c k n o w l e d g e m e n t

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A p p e n d i x

Some commutation relations for powers of pick- and stuff-operators

let be $r \neq s$

$$R_r^k\, t_{rs}^n = \sum_{j=o}^{\min(n,k)} \frac{n!\,k!}{(n-j)!\,j!\,(k-j)!}\, t_{rs}^{n-j}\, R_s^j\, R_r^{k-j} \tag{A1}$$

$$t_{sr}^n\, S_r^k = \sum_{j=o}^{\min(n,k)} \frac{n!\,k!}{(n-j)!\,j!\,(k-j)!}\, S_r^{k-j}\, S_s^j\, t_{sr}^{n-j} \tag{A2}$$

$$R_r^k\, S_s^n = \sum_{j=o}^{\min(n,k)} \frac{n!\,k!}{(n-j)!\,j!\,(k-j)!}\, S_s^{n-j}\, t_{sr}^j\, R_r^{k-j} \tag{A3}$$

$$R_i^k\, S_i^n = \sum_{j=o}^{\min(n,k)} \frac{n!\,k!}{(n-j)!\,j!\,(k-j)!}\, S_i^{n-j} \left[\prod_{s=1}^{j} (t_{ii}-t_{oo}+n+k-j-s)\right] R_i^{k-j} \tag{A4}$$